

# Nonlocal Long Range Orders in 1D Fermionic Systems

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We prove on a prototype Hamiltonian that hidden long range order is always present in the gapped phases of interacting fermionic systems on one dimensional lattices. It is captured by correlation functions of appropriate nonlocal charge and/or spin operators, which remain asymptotically finite. The corresponding microscopic orders are classified. The results are confirmed by DMRG numerical simulation of the phase diagram of the extended Hubbard model, and of a Haldane insulator phase.

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The behavior of strongly correlated electron systems has been widely investigated to understand the physics of several phenomena in condensed matter, ranging from the insulating regime to high- $T_c$  superconductivity. Due to the many degrees of freedom involved, many aspects of the micro- and macroscopic behavior of these systems remain unclear. Recently their simulation by means of ultracold gases of two-component fermionic atoms trapped onto optical lattices has opened new possibilities, leading for instance to the direct observation of the predicted magnetic [1] and Mott insulating (MI) phases [2]. The latter is efficiently modeled by the Hubbard Hamiltonian. In this case, it has been noticed quite recently [3] that in one dimension (1D) it is possible to identify a nonlocal order parameter in the MI phase, which displays long-range order (LRO); a result that is in agreement with Coleman-Hohenberg-Mermin-Wagner theorem [4] since no continuous symmetry of the system has been broken. The discovery envisaged a description of the underlying parity charge order, whose microscopic configurations are depicted below in the second cartoon of Fig.1: the Mott phase consists of a chain of single fermions with up and down spin, where fluctuations of pairs of empty and doubly occupied sites (holons and doublons) are bounded. The behavior is reminiscent of that observed in the insulating regime of a degenerate gas of bosonic atoms [5].

In general, the observation of gapped phases in 1D systems is not believed to be necessarily related to the presence of some type of LRO, since the strong quantum fluctuation are expected to destroy any such order. In this Letter we show that LRO is instead hidden in *every* gapped phase of one dimensional correlated fermionic systems. The result is achieved by means of a general analysis of the bosonization treatment applied on a prototype lattice model Hamiltonian for these systems. We identify in the lattice the nonlocal parity and string operators responsible for the different types of LRO. As a byproduct, both charge and spin excitations turn out to be independently ordered, while local operators intrinsically generate both. It is tempting to conclude that nonlocal operators are “more fundamental” with respect to the usual local ones, at least for the description of

the possible orders in the ground state phase diagram of these systems. To test our results we perform a density matrix renormalization group (DMRG) analysis at half-filling and zero temperature of the standard extended Hubbard case, focusing on the insulating phases.

We start from the general class of lattice model Hamiltonians historically introduced in Ref.[6] to describe the effects of Coulomb repulsion among electrons on their behavior, the standard Hubbard model being the most familiar example. The low energy behavior of these models is described by an effective Hamiltonian  $\mathcal{H}$  obtained by bosonization treatment (see [7] and references therein). Upon neglecting terms of higher scaling dimension (see also [8]),  $\mathcal{H}$  turns out to be the sum of two decoupled sine-Gordon models. Explicitly, we have

$$\mathcal{H} = \sum_{\nu=c,s} \left( H_0^{(\nu)} + \frac{2g_\nu}{(2\pi\alpha)^2} \int dx \cos[q_\nu \sqrt{8} \Phi_\nu(x)] \right), \quad (1)$$

with  $H_0^{(\nu)} = \frac{v_\nu}{2\pi} \int dx [K_\nu (\pi \Pi_\nu)^2 + K_\nu^{-1} (\partial_x \Phi_\nu)^2]$ . Here  $\Phi_\nu$  is the compactified boson describing the charge ( $\nu = c$ ) and spin ( $\nu = s$ ) excitations, with velocity  $v_\nu$ , Gaussian coupling  $K_\nu$  and conjugate momentum  $\Pi_\nu = \partial_x \Theta_\nu / \pi$ ;  $\alpha$  is a cutoff. Moreover, in terms of the standard notation  $g_c \equiv g_{3\perp} \delta_{n,q_c}^{-1}$ , the corresponding term generated from Umklapp processes being non-vanishing only at commensurate fillings  $n = p/q$  ( $p, q$  integer; we assume  $p = 1$ );  $g_s \equiv g_{1\perp}$ , and  $q_c = q$ ,  $q_s = 1$ .

The cosine terms in (1) become irrelevant in the renormalization group (RG) flow equations unless the fields  $\Phi_\nu$  are pinned to fixed values; in this case, the energy is minimized by the choices

$$\sqrt{2}\Phi_\nu = \frac{\pi}{2q_\nu} (2l + 1) \quad g_\nu > 0 \quad (2)$$

$$\sqrt{2}\Phi_\nu = \frac{\pi}{2q_\nu} 2l \quad g_\nu < 0 \quad (3)$$

with  $l \in \mathbb{N} \cup \{0\}$ . A thorough inspection of the RG equations shows that while both choices of locked values for  $\Phi_c$  amount to the opening of a charge gap  $\Delta_c$ , a spin gap  $\Delta_s$  can open only for  $g_s < 0$ , due to the SU(2) spin symmetry of the Hubbard class of Hamiltonians. To resume,

	$q\sqrt{2}\Phi_c$	$\sqrt{2}\Phi_s$	$\Delta_c$	$\Delta_s$	LRO
LL	$u$	$u$	0	0	none
LE	$u$	0	0	open	$O_P^{(s)}$
MI	0	$u$	open	0	$O_P^{(c)}$
HI	$\pi/2$	$u$	open	0	$O_S^{(c)}$
BOW	0	0	open	open	$O_P^{(c)}, O_P^{(s)}$
CDW	$\pi/2$	0	open	open	$O_S^{(c)}, O_P^{(s)}$

Table I: Correspondence between ground state quantum phases and nonlocal operators that manifest LRO.

in all systems described by  $\mathcal{H}$  it is possible, depending on the filling factor and on the relevant interaction terms, to observe up to 6 phases (shown in Table I) each one with different dominant correlations. In most phases the known dominant correlations of two-point local operators decay to zero with distance following a power law, in agreement with bosonization predictions. On the one hand, in charge-density and bond-ordered wave (CDW and BOW respectively) phases – appearing when just on-site and nearest neighbors diagonal Coulomb interactions are present – LRO was identified with the non-vanishing in the asymptotic limit of appropriate two-point correlators of local operators [7]. On the other hand, quite recently it was noticed that for the standard Hubbard model LRO in MI and Luther Emery (LE) liquid phases is described instead by two-points correlators of suitable nonlocal operators [3]. In the present work we extend the idea of nonlocal order to all possible gapped phases listed in Table I for the general Hamiltonian  $\mathcal{H}$ .

First of all, we define for the lattice model the parity and string operators at a given site  $j$  as

$$O_P^{(\nu)}(j) = \prod_{l=1}^j e^{i\pi S_l^{(\nu)}}, \quad O_S^{(\nu)}(j) = S_j^{(\nu)} \prod_{l=1}^{j-1} e^{i\pi S_l^{(\nu)}}, \quad (4)$$

respectively, with  $\nu = c, s$ , and  $S_j^{(c)} = (n_j - 1)$ ,  $S_j^{(s)} = (n_{j\uparrow} - n_{j\downarrow})$ . Here  $n_{j\sigma}$  is the number operator counting the electrons with spin  $\sigma$  ( $\sigma = \uparrow, \downarrow$ ) at site  $j$ , namely  $n_{j\sigma} \equiv c_{j\sigma}^\dagger c_{j\sigma}$ ,  $c_{j\sigma}$  being the operator which annihilates one electron of this type and  $c_{j\sigma}^\dagger$  its Hermitian conjugate; moreover  $n_j \equiv n_{j\uparrow} + n_{j\downarrow}$ . The related two-point correlators  $C_P^{(\nu)}(r) \equiv \langle O_P^{(\nu)}(j) O_P^{(\nu)\dagger}(j+r) \rangle$  (parity correlator), and  $C_S^{(\nu)}(r) \equiv \langle O_S^{(\nu)}(j) O_S^{(\nu)\dagger}(j+r) \rangle$  (string correlator) can be approximated in the continuum limit according to the analysis outlined in Ref. [3, 9], giving

$$C_P^{(\nu)}(x) \approx \langle \cos \sqrt{2}\Phi_\nu(0) \cos \sqrt{2}\Phi_\nu(x) \rangle \quad (5)$$

$$C_S^{(\nu)}(x) \approx \langle \sin \sqrt{2}\Phi_\nu(0) \sin \sqrt{2}\Phi_\nu(x) \rangle, \quad (6)$$

where  $\langle \rangle$  stands for the average evaluated in the ground state. From the above result one can realize that at least one of the parity or string correlators is non-vanishing for  $x \rightarrow \infty$  in every gapped phase. Indeed these take place

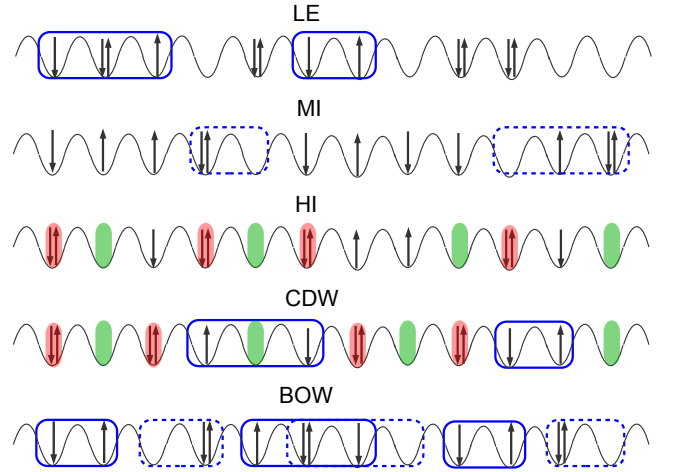


Figure 1: Cartoon picture illustrating how order is maintained in presence of fluctuations. The blue continuous (dashed) lines show the correlated pairs of up-down spin (holon-doublon) allowing  $\langle O_P^{(s)} \rangle$  ( $\langle O_P^{(c)} \rangle$ ) to remain non vanishing. The green and red circles show the alternation of sites occupied by doublons and holons in the chain of single fermions preserving  $\langle O_S^{(c)} \rangle \neq 0$ .

when some  $\Phi_\nu$  is pinned to a fixed values, as shown in Table I. In that case, an order parameter  $\langle O_\alpha^{(\nu)} \rangle$  emerges, since we observe

$$\lim_{x \rightarrow \infty} C_\alpha^{(\nu)}(x) = \langle O_\alpha^{(\nu)} \rangle^2 \equiv C_\alpha^{(\nu)}, \quad \alpha = P, S$$

In Table I, LL stands for the gapless Luttinger Liquid phase, which is the only case without any LRO, as both the bosonic fields  $\Phi_\nu$  are unlocked (we indicate this with  $u$ ). LE is the conducting phase with open spin gap which takes place for  $\Phi_s = 0$ , and is characterized by a nonzero  $\langle O_P^{(s)} \rangle$ . Charge-gapped phase with  $\Delta_s = 0$  can open for a)  $\Phi_c = 0$  (MI), in which case  $\langle O_P^{(c)} \rangle \neq 0$  [3]; b) for  $\Phi_c = \pi/\sqrt{8}$ , that we indicate as Haldane insulator (HI) since it is distinguished by the non-vanishing value of the Haldane-like string order  $\langle O_S^{(c)} \rangle$ . Finally, BOW and CDW phases are fully gapped phases with two non vanishing  $O_\alpha^{(\nu)}$ 's. Notice that only in these latter cases, the two nonlocal order parameters combine to form a true local LRO, namely the BOW and CDW orders that give the name to the corresponding phases [7, 9].

The non-vanishing of the parity and/or string correlators gives further physical insight about the kind of microscopic orders underlying the phases. These are illustrated schematically in Fig.1. At half-filling a non-zero value of the charge (spin) parity correlator implies the formation of bound pairs of holons and doublons (up and down spins) in a background of single electrons (holons and doublons) as it occurs in the MI (LE) phase [3]. Whereas a finite value of the charge (spin) string correlator, no matter the distribution of up and down electrons

(holons and doublons), amounts to a holon (spin up) always followed by a doublon (spin down) site on the holon-doublon (single electrons) sublattice. Not surprisingly, in this latter case the microscopic structure of the phase is quite resemblant of that of the HI observed in the bosonic case. Understanding the microscopic configurations in the different phases also allows insights onto the mechanisms at the basis of the formation of charge and spin gaps. With respect to the perfect MI of singly occupied sites, the Mott charge gap at half-filling is maintained by adding localized pairs formed by a doublon and a holon; whereas the transition to an HI charge gap takes place when the added doublons and holons do organize into a sublattice obtained by ignoring singly occupied sites. The LE case illustrates how an open spin gap, ideally amounting to a configuration with holons and doublons only, is preserved when single electrons are arranged in localized pairs with up and down spins; this observation explains the reason for the dominant character of superconducting correlations in such phase. Finally, combinations of the above possibilities determines the structures of the two fully gapped phases (CDW, and BOW).

In order to support our predictions, we present below a numerical analysis of LRO parameters given by (5), (6) for the insulating phases of the extended Hubbard model at half-filling in case of repulsive interactions. In this case the lattice Hamiltonian reads

$$H = -t \sum_{j\sigma} (c_{j\sigma}^\dagger c_{j+1,\sigma} + \text{H.c.}) + U \sum_j n_{j\uparrow} n_{j\downarrow} + V \sum_j n_j n_{j+1} \quad (7)$$

where  $U$  and  $V$  represent the diagonal on-site and neighbor sites contribution of the interaction potential and we fix the energy scale with  $t = 1$ . Such model, which is of fundamental relevance in condensed matter and in the younger field of ultracold systems, has been thoroughly explored in the literature (see [7, 10, 11] and references therein). Indeed, recent experiments with Fermi gas of magnetic atoms [12] or polar molecules [13] allow to quantitatively simulate the Hamiltonian (7) where both the interactions parameters can be tuned by changing the direction of the dipoles with external fields or the transverse frequency of the laser used to create the lattice. In particular, we want to explore at half-filling the regime of positive values of  $U$  and  $V$ , for which the phase diagram amounts to three different insulating phases.

The analysis is performed using a DMRG algorithm on finite size chains with periodic boundary conditions. We have chosen to consider small system sizes, from  $L = 12$  to 48, with up to 1600 DMRG states and six sweeps in order to have a good precision on our quantities.

We expect that the parity and string operators introduced above should work as order parameters for the three insulating phases. More precisely (see also Table I),

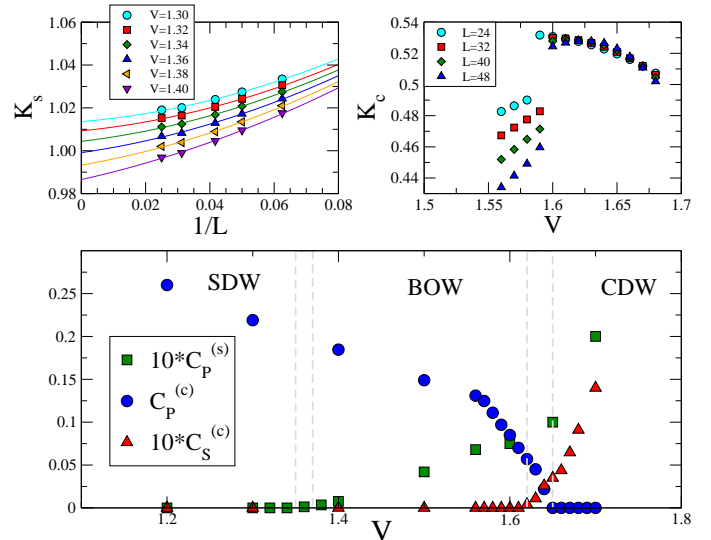


Figure 2: Lower panel: Nonlocal order parameters  $C_\alpha^{(\nu)}(L/2)$  for  $L \rightarrow \infty$  for the three insulating phases of the extended Hubbard model (7). The vertical dashed lines locate the critical points with the uncertainty determined from the numerical analysis reported in the upper panels. Upper-left panel:  $K_s$  as a function of  $1/L$  for different values of  $V$ . The continuous lines are second order polynomial functions used to extrapolate the TDL. Upper-right panel:  $K_c$  as a function of  $V$  for different system sizes.

the asymptotic value of  $\langle O_P^{(c)} \rangle$  should be the only non-vanishing parameter for the MI phase; whereas in the BOW phase also  $\langle O_P^{(s)} \rangle$  should become different from zero at the MI-BOW transition. Finally, at the BOW-CDW transition  $\langle O_S^{(c)} \rangle$  should become different from zero, while  $\langle O_P^{(c)} \rangle$  becomes vanishing.

We have calculated  $C_P^{(\nu)}(r) = \langle \exp(\sum_{l=j}^{j+r} i\pi S_l^{(\nu)}) \rangle$  and  $C_S^{(\nu)}(r) = \langle S_j^{(\nu)} \exp(\sum_{l=j+1}^{j+r-1} i\pi S_l^{(\nu)}) S_{j+r}^{(\nu)} \rangle$ , and their asymptotic values have been evaluated at the mid point  $r = L/2$ , upon an extrapolation in the thermodynamic limit (TDL)  $L \rightarrow \infty$ . Special care must be paid in separating the uniform and staggered parts of the parity operator, since the relation  $C_P^{(c)}(r) = (-1)^r C_P^{(s)}(r)$  holds. Fig.2 collects our numerical results, showing a clear evidence of the expected behavior.

Our findings can be compared with those obtained in [14] by considering the expectation value of a different non-local operator, namely the exponential position operator  $z_L$ . Since in bosonization analysis such value takes the form  $\langle \cos \sqrt{8}\Phi_c \rangle$ , it is different from zero for both pinned values of  $\Phi_c$  allowed in an insulating phase, hence vanishing only at the conducting point where the BOW-CDW transition takes place [15].

To enforce our analysis we also computed the Luttinger constants  $K_\nu$  defined as  $K_\nu \sim \lim_{q \rightarrow 0} \pi \mathcal{S}_\nu(q)/q$ , with  $\mathcal{S}_\nu(q) = \frac{1}{L} \sum_{kl} e^{iq(k-l)} (\langle S_{k,z}^\nu S_{l,z}^\nu \rangle - \langle S_{k,z}^\nu \rangle \langle S_{l,z}^\nu \rangle)$  in the TDL. These give precise information regarding the pres-

ence of gaps [9]. In particular the SDW-BOW belongs to the Berezinskii-Kosterlitz-Thouless (BTK) universality class since a spin gap takes place entering in the fully gapped BOW phase, while maintaining a full rotational spin symmetry. The Luttinger theory predicts  $K_s = 1$  in the gapless and  $K_s = 0$  in the gapped phase. Numerically it is hard task to get exactly these two values since in the gapless phase logarithmic corrections affect the results, while in the gapped region really large system sizes are necessary in order to get  $K_s = 0$ . It is customary to locate the transition point where  $K_s$  takes values smaller than 1 in the TDL. As shown in Fig.2, the transition point obtained in this way is in good agreement with the one predicted by  $O_P^{(s)}$ . The BOW-CDW transition requires particular care since its nature can be either second or first order, depending on the value of  $U$ . Here we consider the region  $U < 4$  where the transition is known to be second order. As shown in [11], while the two phases are fully gapped, due to the competition between the onsite and nearest-neighbor interactions the charge gap is minimal at the transition point, where it takes the value 0. Hence the theory predicts a Luttinger parameter  $K_c \neq 0$  only at the gapless point and  $K_c = 0$  elsewhere. In Fig.2 we show that  $K_c$  develops a peak slightly dependent on the system size and extrapolations in the TDL confirm the transition obtained by looking the vanishing value of  $\langle O_P^{(c)} \rangle$  and the finite value of  $\langle O_S^{(c)} \rangle$ .

The scenario of Table I is completed by the identification of the HI phase, where only  $O_S^{(c)}$  is predicted to have finite LRO. The ground state phase diagram of the model (7) does not show such a phase [16]. Nevertheless, in Refs.[7, 17] a charge gapped phase corresponding to the pinned value  $\Phi_c = \pi/\sqrt{8}$  was identified by adding to the Hamiltonian (7) further correlated hopping terms of the form  $X \sum_{\langle ij \rangle \sigma} (c_{i\sigma}^\dagger c_{j\sigma} + \text{H.c.})(n_{i\bar{\sigma}} - n_{j\bar{\sigma}})^2$  for an appropriate range of values of  $U$  and  $V$ . Such phase was denoted as bond-spin-density wave (BSDW), albeit the spin order cannot show LRO due to the unbroken  $SU(2)$  symmetry. On the basis of our analysis, since  $\Phi_s$  is unpinned, we expect such a phase to exhibit the searched HI order. We have numerically estimated the nonlocal correlators  $C_\alpha^{(\nu)}(L/2)$  at various  $L$  in a single point inside the phase ( $X = 0.25, U = 1, V = 0.5$ ). The results shown in Fig.3 demonstrate that, within the numerical errors, in the asymptotic limit (and in the TDL) the only operator that supports LRO is  $O_S^{(c)}$ , as expected.

We notice that further nonlocal orders may appear in fermionic systems as a consequence of reduced symmetries. For instance, relaxing the  $SU(2)$  spin symmetry to  $U(1) \times \mathbb{Z}_2$ , may allow for the appearance of the value  $\Phi_s = \pi/\sqrt{8}$  in Eq.(1), giving rise to Haldane-like correlations in the  $z$ -component of the spin. Further breaking of the two  $U(1)$  symmetries related to particle number conservation and spin rotation in the  $xy$  plane, open the way to a pinning of the dual fields  $\Theta_c$  and  $\Theta_s$ , re-

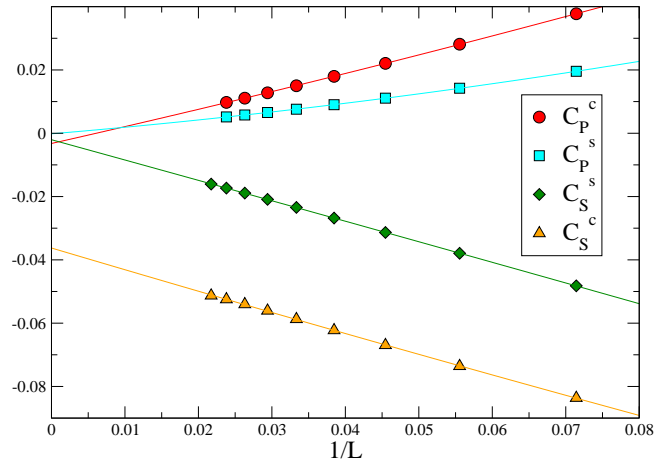


Figure 3: Nonlocal LRO in the Haldane insulator phase, at  $U = 1, V = 0.5$  for the model (7) plus a correlated hopping term with  $X = 0.25$  (see text). As predicted from Table I all correlation functions  $C_\alpha^{(\nu)}(L)$  vanish asymptotically except for  $C_S^{(c)}$ . Continuous lines represent nonlinear fits for estimating the asymptotic limit.

spectively. As a consequence the correlators related to the operators  $\cos(\sqrt{2}\Theta_\nu)$  and  $\sin(\sqrt{2}\Theta_\nu)$  are also finite, thus generating a transverse Haldane-type order, similarly to what happens in spin-1 chains [18] or in the bosonic case [19, 20]. This simple argument suggests that for observing a Haldane order in all directions in fermionic systems it is imperative to extend interacting models like Eq.(7) by including pair creation terms of the kind  $\sum_{j\sigma} (c_{j\sigma}^\dagger c_{j+1,-\sigma}^\dagger + \text{H.c.})$ . Such analysis represents an intriguing topic *per se* that goes beyond the scopes of the present work and will be addressed elsewhere.

In this Letter, we have proved that LRO underlies all the gapped phases of a large class of lattice model Hamiltonians, describing 1D correlated fermionic systems. The order is captured by appropriate two-points nonlocal correlators, describing separately the charge and spin excitations, which remain asymptotically finite in such phases. The observation allows to reconstruct the types of microscopic orders underlying the phases. In particular, it can be applied to discuss the phase diagram of the extended Hubbard model at half-filling, as well as the presence of an Haldane insulating phase in fermionic systems. Our results give precise indications as for the nonlocal quantities to detect in experiments with fermionic trapped dipolar atoms. These are directly accessible to experimental detection in optical lattices via single site resolution imaging with the available techniques [5].

The generality of the analysis here described suggests the presence of a universal mechanism extendable to any system in 1D, stating the presence of appropriate nonlocal LRO in every phase that shows a gap in the excitation spectrum. A further interesting topic which is still under debate concerns the relationship between the role

of nonlocal order, topological phases, and long distance entanglement [21] in 1D.

Regarding the possible presence of the discussed type of nonlocal orders in higher dimension, one should stress the different nature of the parity versus string order parameters, as depicted also in the cartoons of Fig.1. Indeed, the parity LRO is preserved by adding pairs of particles that locally break the order, and this can be done in arbitrary dimension. At variance, the string order requires that the added particles organise into a nonlocal 1D structure, a feature that seems difficult to generalise to higher dimension. With this in mind, it is reasonable to expect that phases with just parity order parameters (MI, LE, and BOW) could describe the physics of two-dimensional systems as well. This is consistent with very recent results on the relevance of parity correlator in the MI phase of the 2D Bose-Hubbard model [22], as well as with findings on backflow correlations in the 2D Hubbard model [23], which emphasize the role of holon-doublon attraction in the MI phase.

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